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A bond percolation critical probability determination based on the star–triangle transformation†

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Abstract. The bond percolation critical probability of a planar graph with square and triangular faces, obtained by inserting a diagonal in every other face of the square lattice, is the root of $1-p-6p^2+6p^3-p^5=0$ in $(0, 1)$, which is approximately 0.404 518. The proof uses the star–triangle transformation to determine the parameter value for which the percolative behaviour of the lattice and its dual lattice are identical.

1. Introduction

The classical problem of mathematical percolation theory is the rigorous determination of critical percolation probabilities. A heuristic method due to Sykes and Essam derived the conjectured value of $\frac{1}{2}$ for the square lattice bond model and triangular lattice site model, based on self-duality and self-matching properties of the lattices respectively. Following pioneering work by Seymour and Welsh (1978) and Russo (1978), Kesten (1980, 1982) rigorously verified these values. Sykes and Essam's method, plus additional reasoning using the star–triangle transformation, yielded the values $2 \sin(\pi/18)$ and $1-2 \sin(\pi/18)$ for the triangular and hexagonal lattice bond models, respectively. Wierman (1981) verified these values. However, counter-examples have been given by Van den Berg (1981) and Wierman (1984) to other claims relating to the Sykes and Essam method.

Kesten (1982) supplied results on critical probabilities and surfaces in the context of multiparameter site percolation models on two-dimensional periodic graphs with one axis of symmetry. This paper applies Kesten's principal theorems and the star-triangle transformation to evaluate the bond percolation critical probabilities of the square lattice with diagonal bonds inserted in every other face and its dual lattice. The graphs, shown in figure 1, have critical probabilities $p_0 \simeq 0.404\,518$, the root of $1-p-6p^2+6p^3-p^5$ in $[0, 1]$, and $1-p_0$ respectively. Definitions and discussion of Kesten's results are presented in § 2, with the derivation of the critical probability values provided in § 3.

2. Background

In a bond percolation model on a graph $\mathscr{G}$, each bond is open with probability p, $0 \leq p \leq 1$, independently of all other bonds. The probability measure and expectation

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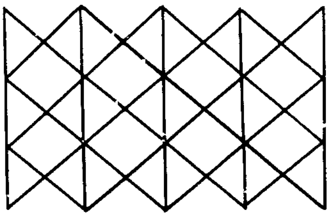
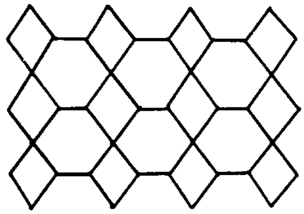


(*a*)



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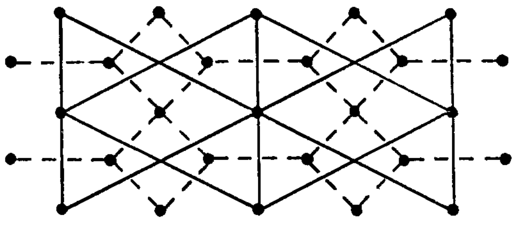

(*c*)

Figure 1. (*a*) Shows the square lattice with diagonals inserted in every other face. (*b*) is its dual graph. (*c*) illustrates the duality relationship.

operator corresponding to parameter value p are denoted by P_p and E_p respectively.

A *path* on $\mathscr{G}$ is an alternating sequence $r=(s_0, b_1, s_1, b_2, \ldots, s_{n-1}, b_n, s_n)$ of sites $\{s_i\}$ and $\{b_i\}$ such that b_i has s_{i-1} and s_i as endpoints for each $i=1, \ldots, n$. A path r is a circuit if $s_0=s_n$. A path is open (closed) if all its bonds are open (closed). For any site s, the open cluster containing s, denoted W_s, is the set of all bonds which are in any open path containing the site s. Let $\# W_s$ denote the number of bonds in the open cluster containing s.

Several different definitions of critical probability have been proposed. The cluster size critical probability is

$$p_{\mathrm{H}}(\mathscr{G})=\inf\{p\in[0,1]\colon P_p(\# W_s=+\infty)>0\}.$$

The expected cluster size critical probability is

$$p_{\mathrm{T}}(\mathscr{G})=\inf\{p\in[0,1]\colon E_p(\# W_s)=\infty\}.$$

If $\mathscr{G}$ is connected, the values of p_{T} and p_{H} are independent of the choice of site s. For any $\mathscr{G}$, $p_{\mathrm{T}}(\mathscr{G})\leqslant p_{\mathrm{H}}(\mathscr{G})$.

A crucial development in the rigorous evaluation of critical probabilities was the introduction of the sponge-crossing critical probability by Seymour and Welsh (1978). An i-crossing of the rectangle $R=[m_1, n_1]\times[m_2, n_2]=\{x=(x(1), x(2))\colon m_i\leqslant x(i)\leqslant n_i, n_i, i=1,2\}$ is a path $(s_0, b_1, \ldots, s_n)$ such that $s_0(i)\leqslant m_i$, $s_n(i)\geqslant n_i$, b_1 intersects $\{x(i)=m_i\}\cap R$, b_n intersects $\{x(i)\}\cap R$, and $s_1, b_2, \ldots, s_{n-1}$ are contained in the interior of R. The crossing probability in the ith direction of $[m_1, n_1]\times[m_2, n_2]$ is $\sigma(m_1, n_1; m_2, n_2; i, p)=P_p\{\exists$ an open i-crossing of $[m_1, n_1]\times[m_2, n_2]\}$.

Kesten's general results all are proved in the setting of periodic graphs. A periodic graph in the plane is a connected graph which contains no loops (i.e. each bond has two distinct endpoints), has at most $z<\infty$ bonds incident to any site, and such that all bonds have finite length, every compact set intersects only finitely many bonds, and the sets of sites and bonds are invariant under translation by any vector with integer components. For periodic graphs, the crossing probabilities essentially do not depend on the location of the rectangle, but only on the length and width. The sponge-crossing critical probability is defined for a periodic graph $\mathscr{G}$ by $P_S(\mathscr{G}) = \inf\{p \in [0, 1]: \limsup_{n\to\infty} \sigma(0, n; 0, 3n; 1, p) \vee \sigma(0, 3n; 0, n; 2, p) > 0\}$. (The original definition by Seymour and Welsh used crossings of squares. The present definition is due to Kesten (1981).) One of the few dimension-free results in percolation is that $p_S(\mathscr{G}) = p_T(\mathscr{G})$ for a periodic graph $\mathscr{G}$ of any dimension (see Kesten 1981 or 1982).

For a planar graph $\mathscr{G}$, the dual graph $\mathscr{G}^*$ is constructed by placing a site of $\mathscr{G}^*$ in each face of $\mathscr{G}$, and connecting two sites of $\mathscr{G}^*$ if the corresponding faces share a common edge. Thus there is a one-to-one correspondence between edges of $\mathscr{G}$ and edges of $\mathscr{G}^*$, and $(\mathscr{G}^*)^* = \mathscr{G}$. If $\mathscr{G}$ is periodic, then one may construct $\mathscr{G}^*$ to be periodic also.

The one-to-one correspondence allows a percolation model on $\mathscr{G}$ to induce a dual percolation model on $\mathscr{G}^*$. Each bond of $\mathscr{G}^*$ is open (closed) if and only if the corresponding bond of $\mathscr{G}$ is open (closed). An open cluster in one lattice is bounded by a closed circuit in the dual lattice. Crossing probabilities in the dual graph $\mathscr{G}^*$ will be denoted by

$$\sigma^*(m_1, n_1; m_2, n_2; i, p) = P_p(\exists \text{ a closed } i\text{-crossing of } [m_1, n_1]\times[m_2, n_2]).$$

Kesten's principal theorems are formulated for two-dimensional periodic graphs with one axis of symmetry. They provide conditions on the probability parameter p, in terms of crossing probabilities on $\mathscr{G}$ and $\mathscr{G}^*$, which can be used to verify that a proposed value p_0 is the critical probability $p_H(\mathscr{G}) = p_T(\mathscr{G}) = p_S(\mathscr{G})$, and $p_H(\mathscr{G}) + p_H(\mathscr{G}^*) = 1$. One condition requires a relation between open crossing probabilities on $\mathscr{G}$ and closed crossing probabilities on $\mathscr{G}^*$ in the same direction, while another requires a relation between horizontal open crossings and vertical open crossings on each of $\mathscr{G}$ and $\mathscr{G}^*$. For detailed statements of these results, see Kesten (1982, ch 3). We now state a very specialised corollary of these results, which will suffice for the graphs in this paper.

Lemma. Consider the bond percolation model on a planar periodic graph $\mathscr{G}$ with one axis of symmetry. Suppose there exists $p_0 \in [0, 1]$, $0 < c_i \leq C_i < \infty$ and $a_{ij}, b_{ij} \in \mathbb{R}$ such that

$$\begin{aligned} c_i\sigma(m_1 + a_{11}, n_1 + a_{12}; m_2 + a_{21}, n_2 + a_{22}; i, p_0) \\ \leq \sigma^*(m_1 + b_{11}, n_1 + b_{12}; m_2 + b_{21}, n_2 + b_{22}; i, p_0) \\ \leq C_i\sigma(m_1 + a_{11}, n_1 + a_{12}; m_2 + a_{21}, n_2 + a_{22}; i, p_0) \end{aligned}$$

for all $m_1, n_1, m_2, n_2 \in \mathbb{Z}$, and $i = 1, 2$. Then

$$p_H(\mathscr{G}) = p_T(\mathscr{G}) = p_S(\mathscr{G}) = p_0$$

and $p_H(\mathscr{G}^*) = p_T(\mathscr{G}^*) = p_S(\mathscr{G}^*) = 1 - p_0$.

3. Derivation of critical probabilities

Construct an infinite graph $\mathscr{H}$ imbedded in $\mathbb{R}^2$ as follows. For each pair of integers (i, j), locate a site of $\mathscr{H}$ at each of the points (i, j) and $(i+\frac{1}{2}, j+\frac{1}{2})$. Connect the site at (i, j) by a bond to each of the sites $(i, j-1)$, $(i, j+1)$, and $(i\pm\frac{1}{2}, j\pm\frac{1}{2})$ (see figure 1). Notice that, upon rotation by 45°, $\mathscr{H}$ may be viewed as a square lattice with a diagonal inserted in every other face. The dual lattice $\mathscr{H}^*$ may be constructed as a periodic lattice by locating its sites at $(i+\frac{1}{2}, j)$, $(i+\frac{1}{4}, j+\frac{1}{2})$ and $(i+\frac{3}{4}, j+\frac{1}{2})$ for each pair of integers (i, j).

The standard bond percolation model on $\mathscr{H}$ may be transformed into an equivalent bond percolation model on a related graph. Let $\mathscr{H}_1$ denote the graph obtained from $\mathscr{H}$ by replacing each vertical bond of $\mathscr{H}$ by two vertical bonds between the same pair of sites. If each of these vertical bonds on $\mathscr{H}_1$ is open with probability $1-(1-p)^{1/2}$, in the resulting model the probability that (i, j) and $(i, j+1)$ are connected by one or both bonds possible in $\mathscr{H}_1$ being open is p. Under this transformation the subgraph of $\mathscr{H}$ contained in the rectangle $[\frac{1}{2}, m+\frac{1}{2}]\times[0, n]$ shown in figure 2(a) is transformed into the subgraph of $\mathscr{H}_1$ shown in figure 2(b).

The star–triangle transformation may be applied to $\mathscr{H}_1$. Each square $(i, j)+[0, 1]\times[0, 1]$ contains two triangular faces of $\mathscr{H}_1$. By Sykes and Essam (1964), if the sides of a triangle are open with probabilities a, b and c, and the opposite bonds on a super-imposed 'star' are open with probabilities a, b and c respectively, the probabilities of connections between the vertices of the triangle with open bonds on the triangle and closed bonds on the star are identical if $1-a-b-c+abc=0$ (see figure 3). In the model on $\mathscr{H}_1$, this condition is satisfied if

$$1-2p-[1-(1-p)^{1/2}]+p^2[1-(1-p)^{1/2}]=0$$

which implies that

$$1-p-6p^2+6p^3-p^5=0. \tag{1}$$

Let p_0 denote the solution of (1) in [0, 1], which is approximately 0.404 518. If $p=p_0$, applying the star–triangle transformation to each triangular face of $\mathscr{H}_1$ (locating the center site of the star at $(i+\frac{1}{4}, j+\frac{1}{2})$ or $(i+\frac{3}{4}, j+\frac{1}{2})$), we obtain an equivalent bond percolation model on the graph $\mathscr{H}_2$ shown in figure 2(c). In fact, $\mathscr{H}_2$ is the dual graph $\mathscr{H}_1^*$. The percolative behaviour of open bonds on the subgraph of $\mathscr{H}_1$ contained in $[\frac{1}{2}, m+\frac{1}{2}]\times[0, n]$ is equivalent to that of closed bonds on the subgraph of $\mathscr{H}_2$ shown in figure 2(c), in which each horizontal bond is closed with probability $(1-p_0)^{1/2}$ and each diagonal is closed with probability $1-p_0$.

Replace each pair of adjacent horizontal bonds in $\mathscr{H}_2$, each being closed with probability $(1-p_0)^{1/2}$, by a single horizontal bond which is closed with probability $1-p_0$. This transformation preserves equivalence of connectivities by closed bonds in

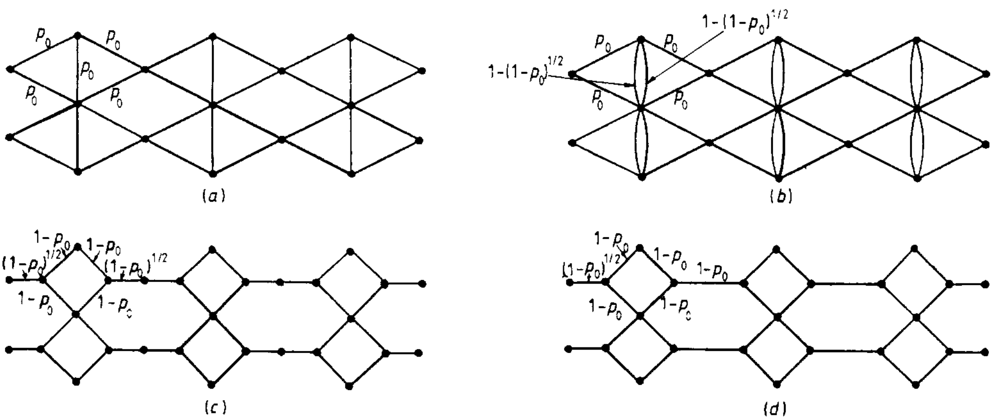

Figure 2. Open bond probabilities are indicated for bonds on the upper left in (a) and (b); closed bond probabilities in (c) and (d).

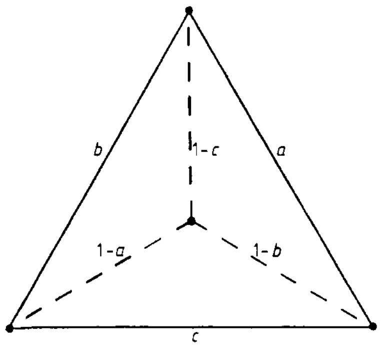

Figure 3. Star-triangle transformations.

the previous model on $\mathscr{H}_2$ and closed bonds in the resulting model on a translation $\mathscr{H}_3$ of the dual $\mathscr{H}^*$ of the original graph. (Note that in a standard bond model where connectivities through open bonds are considered, the proper probability that the replacement bond is closed would be $1-[1-(1-p_0)^{1/2}]^2$. Since we are considering connections through closed bonds, for the replacement bond to be closed we require both replaced bonds to be closed.) When applying this transformation to the rectangular region $[\frac{1}{2}, m+\frac{1}{2}]\times[0, n]$, the pairs of adjacent horizontal bonds in the interior of the region may be replaced as above. However, single horizontal bonds appear on the left and right edges. Denote the resulting graph by $\mathscr{R}$. In the resulting equivalent model on $\mathscr{R}$, shown in figure 2(*d*), let $\sigma^*(\mathscr{R}; i, p)$ denote the probability that there exists a closed crossing of $\mathscr{R}$ in the *i*th direction when the parameter is *p*, for $i=1,2$. By the preceding sequence of equivalences, for $i=1,2$,

$$\sigma^*(\mathscr{R}; i, p_0)=\sigma(\tfrac{1}{2}, m+\tfrac{1}{2}; 0, n; i, p_0).$$

Expand $\mathscr{R}$ to obtain a graph $\mathscr{R}_1$ by adding bonds from $(\frac{1}{4}, j+\frac{1}{2})$ to $(\frac{1}{2}, j+\frac{1}{2})$ and from $(m+\frac{1}{2}, j+\frac{1}{2})$ to $(m+\frac{3}{4}, j+\frac{1}{2})$ for each $0\leq i\leq n-1$. Let each of these bonds be closed with probability $(1-p_0)^{1/2}$, and $\sigma^*(\mathscr{R}_1; i, p)$ denote the probability that there exists a closed crossing of $\mathscr{R}_1$ in the *i*th direction when the parameter *p*. The addition of bonds to $\mathscr{R}$ does not affect the probability of vertical crossings, so

$$\sigma^*(\mathscr{R}_1; 2, p_0)=\sigma(\tfrac{1}{2}, m+\tfrac{1}{2}; 0, n; 2, p_0).$$

A horizontal closed crossing of $\mathscr{R}_1$ contains a horizontal closed crossing of $\mathscr{R}$, so

$$\sigma^*(\mathscr{R}_1; 1, p_0)\leq\sigma^*(\mathscr{R}; 1, p_0)=\sigma(\tfrac{1}{2}, m+\tfrac{1}{2}; 0, n; 1, p_0).$$

Since a horizontal closed crossing of $\mathscr{R}$ may be extended to a horizontal closed crossing of $\mathscr{R}_1$ by addition of a closed bond on each end, we also have

$$\sigma^*(\mathscr{R}_1; 1, p_0)\geq(1-p_0)\sigma^*(\mathscr{R}_1; 1, p_0)=(1-p_0)\sigma(\tfrac{1}{2}, m+\tfrac{1}{2}; 0, n; 1, p_0).$$

Replace each pair of adjacent horizontal bonds on the left and right sides of $\mathscr{R}_1$ by a single horizontal bond which is closed with probability $1-p_0$, as in transforming $\mathscr{H}_2$ into $\mathscr{H}_3$. This produces a graph which is a translation of the subgraph of $\mathscr{H}^*$ contained in $[-\frac{1}{4}, m+\frac{1}{4}]\times[0, n]$, with the dual percolation model for $\mathscr{H}$. Therefore

$$(1-p_0)\sigma(\tfrac{1}{2}, m+\tfrac{1}{2}; 0, n; 1, p_0)\leq\sigma^*(-\tfrac{1}{4}, m+\tfrac{1}{4}; 0, n; 1, p_0)\leq\sigma(\tfrac{1}{2}, m+\tfrac{1}{2}; 0, n; 1, p_0)$$

and

$$\sigma^*(-\tfrac{1}{4}, m+\tfrac{1}{4}; 0, n; 2, p_0)=\sigma(\tfrac{1}{2}, m+\tfrac{1}{2}; 0, n; 2, p_0),$$

for any positive integers m and n. Thus, the hypotheses of lemma 1 are satisfied, so we conclude that

$$p_S(\mathscr{H}) = p_T(\mathscr{H}) = p_H(\mathscr{H}) = p_0 \simeq 0.404\,518$$

and

$$p_S(\mathscr{H}^*) = p_T(\mathscr{H}^*) = p_H(\mathscr{H}^*) = 1 - p_0 \simeq 0.595\,482.$$

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